Online Technical Appendix for "Browse or Experience"

Specifics of Consumer Learning:

1. Learning about Attributes:

In the interpretation of consumer learning while owning or not owning the product, we can consider that consumers are leaning about attributes of equal importance in each instant in time, with the overall utility being the sum of the deviation to the mean of each attribute's contribution. See Branco, Sun, and Villas-Boas (2012) as an example. When not owning the product the expected flow utility if the consumer were to buy the product is the sum of the deviation to the mean of each of the known attribute's contribution. When owning the product the experienced flow utility is also assumed to be the sum of the deviation to the mean of each of the known attribute's contribution. This is consistent with the idea that when experiencing a product, the consumer does not learn perfectly the flow utility that can be obtained with that product.[1] If the consumer learns more attributes per unit of time when owning than when not owning the product, we have $s^2 > \sigma^2$.

Consider T as the mass of attributes. The main text presents the case of $T \to \infty$.

Alternatively, we could have T distributed exponentially with parameter ψ, with the consumers not knowing T. In that case, if the consumer does not own the product and information on all the number of attributes has been obtained, the consumer gets a present value of utilities $W_T(x) = \max[0, \frac{x-\lambda P}{r}]$. If the consumer owns the product all information on all attributes have been obtained, the consumer gets a present value of utilities of

$$V_T(x) = \frac{\max[0, x] + \lambda W_T(x)}{r + \lambda}. \tag{1}$$

When the consumer owns the product and $x > \overline{x}$ and the consumer has not yet checked all attributes, we have that (**??**) is replaced with

$$\begin{aligned} V(x) &= x\,dt + e^{-r\,dt}(1-\psi\,dt)\left[\lambda\,dt\{E[V(x+dx)] - P\} + (1-\lambda\,dt)E[V(x+dx)]\right] \\ &\quad + e^{-r\,dt}\psi\,dt\left[\lambda\,dt W_T(x) + (1-\lambda\,dt)V_T(x)\right]. \end{aligned} \tag{2}$$

When the consumer owns the product and $x \in (0, \overline{x})$, we have that (3) and (4) in the

[1] Alternatively, we could consider that the experienced utility is only fully realized after the consumer leaves the market permanently.

paper are replaced with

$$\begin{aligned}\widetilde{V}(x) &= \max[0, x\,dt] + e^{-r\,dt}(1-\psi\,dt)[\lambda\,dt W(x) + (1-\lambda\,dt)E\widetilde{V}(x+dx)] \\ &\quad + e^{-r\,dt}\psi\,dt[\lambda\,dt W_T(x) + (1-\lambda\,dt)V_T(x)]. \end{aligned} \tag{3}$$

We can then conduct the analysis as in the main text and obtain the (stationary) threshold $\overline{x}$, which is now a function of ψ as well. The case presented in the main text is the case in which $\psi \to 0$.

2. Signals of Product Value:

Consider an alternative model, where the true value of the product, $\widehat{x}$, evolves over time, as

$$d\widehat{x}_t = \widetilde{\sigma} dW_t \tag{4}$$

where W_t is a Wiener process. The decision maker observes a noisy signal S_t^x at time t, which follows

$$S_t^x = \widehat{x}_t + \widetilde{s}V_t \tag{5}$$

or

$$dS_t^x = d\widehat{x}_t + \widetilde{s}dV_t \tag{6}$$

where V_t is a Wiener process. That is, the signal is only on the change of $\widehat{x}$.

For simplicity, assume that $\widehat{x}_0$ is known. Then we have $\widehat{x}_t \sim \mathcal{N}(\widehat{x}_0, \widetilde{\sigma}^2 t)$, and noise $S_t^x - \widehat{x}_t \sim \mathcal{N}(0, \widetilde{s}^2 t)$. The posterior mean x_t is

$$\begin{aligned} x_t &= \frac{\frac{\widehat{x}_0}{\widetilde{\sigma}^2 t} + \frac{N_t}{\widetilde{s}^2 t}}{\frac{1}{\widetilde{\sigma}^2 t} + \frac{1}{\widetilde{s}^2 t}} && (7) \\ &= \frac{\widetilde{s}^2}{\widetilde{\sigma}^2 + \widetilde{s}^2}\widehat{x}_0 + \frac{\widetilde{\sigma}^2}{\widetilde{s}^2 + \widetilde{\sigma}^2} N_t && (8) \end{aligned}$$

Because $S_t^x \sim \mathcal{N}(\widehat{x}_t, \widetilde{s}^2 t)$ and $\widehat{x}_t \sim \mathcal{N}(\widehat{x}_0, \widetilde{\sigma}^2 t)$, we have $X_t^x \sim \mathcal{N}(\widehat{x}_0, (\widetilde{\sigma}^2 + \widetilde{s}^2)t)$. Thus

$$\mathbb{E}[(x_t - \widehat{x}_0)^2] = \frac{\widetilde{\sigma}^4}{\widetilde{\sigma}^2 + \widetilde{s}^2} t \tag{9}$$

so x_t evolves with a constant variance that is decreasing in $\widetilde{s}^2$. With infinite noise, x_t does not update, and with no noise, x_t updates with variance $\widetilde{\sigma}^2$, which is intuitive.

If $\widetilde{s}_n^2$ is variance of the noise when not owning the product, and $\widetilde{s}_o$ is the variance of the noise when owning the product, we can then obtain the representation as in the main text with $\sigma^2 = \frac{\widetilde{\sigma}^4}{\widetilde{\sigma}^2+\widetilde{s}_n^2}$ and $s^2 = \frac{\widetilde{\sigma}^4}{\widetilde{\sigma}^2+\widetilde{s}_o^2}$. Note that a more informative signal leads to a higher variance on x_t. If the consumer gets more information when owning the product than not owning the product, or $\widetilde{s}_o \leq \widetilde{s}_n$, then we have $s^2 \geq \sigma^2$.

Still another variation could be the case in which the true value $\widehat{x}$ can take only two values, $\{-1, 1\}$, and both search and experience leads to a continuous evolution of the consumer belief about whether the consumer is facing the good or bad product. Although tractable, such a model leads to a more complex analysis than the one presented in the main text.

Derivation of $\overline{x}$ in the Case of $\overline{x} > 0$:

From the differential equation on W(x), and using $\lim_{x\to-\infty} W(x) = 0$, one obtains

$$W(x) = A_1 e^{\mu x} \tag{10}$$

where $\mu = \sqrt{2r/\sigma^2}$, and A_1 is a constant to be determined.[2]

Using Itô's Lemma in (2) in the paper one obtains $rV(x) = x - \lambda P + V''(x)\frac{s^2}{2}$. Note that $\lim_{x\to\infty}[V(x) - (x-\lambda P)/r] = 0$, as when the current utility goes to infinity, the consumer is always buying the product when it breaks down, which generates an expected utility of $(x - \lambda P)/r$. Using this when solving the differential equation on $V(x)$, one obtains

$$V(x) = A_2 e^{-\widetilde{\mu}x} + \frac{x - \lambda P}{r}, \tag{11}$$

where $\widetilde{\mu} = \sqrt{2r/s^2}$, and A_2 is a constant to be determined.

Using Itô's Lemma on (3) in the paper, and solving the resulting differential equation, this yields

$$\widetilde{V}(x) = A_3 e^{\widehat{\mu}x} + A_4 e^{-\widehat{\mu}x} + \frac{x}{r+\lambda} + \frac{\lambda A_1}{r(1 - s^2/\sigma^2) + \lambda} e^{\mu x}, \tag{12}$$

where $\widehat{\mu} = \sqrt{2(r+\lambda)/s^2}$, and A_3 and A_4 are constants to be determined.

Similarly, from (4) in the paper, and using the fact that the expected utility when owning the product goes to zero when the current utility of using the product approaches negative infinity, one obtains

$$\widetilde{V}(x) = A_5 e^{\widehat{\mu}x} + \frac{\lambda A_1}{r(1 - s^2/\sigma^2) + \lambda} e^{\mu x}, \tag{13}$$

[2]Note that the general solution of the differential equation $rW(x) = W''(x)\frac{\sigma^2}{2}$ is $W(x) = A_1 e^{\mu x} + \widetilde{A}_1 e^{-\mu x}$, where A_1 and $\widetilde{A}_1$ are constants. The condition $\lim_{x\to-\infty} W(x) = 0$ then yields $\widetilde{A}_1 = 0$. Similar derivations are also used in the remainder of the paper when appropriate.

where A_5 is a constant to be determined.

Value matching and smooth pasting at both $\overline{x}$ and 0, $W(\overline{x}) = V(\overline{x}) - P, W'(\overline{x}) = V'(\overline{x}), V(\overline{x}) = \widetilde{V}(\overline{x}), V'(\overline{x}) = \widetilde{V}'(\overline{x}), \widetilde{V}(0^+) = \widetilde{V}(0^-)$, and $\widetilde{V}'(0^+) = \widetilde{V}'(0^-)$ yields

$$
\begin{aligned}
A_1\overline{X} &= \frac{A_2}{\widetilde{X}} + \frac{\overline{x} - \lambda P}{r} - P && (14)\\
\mu A_1\overline{X} &= -\frac{\widetilde{\mu}A_2}{\widetilde{X}} + \frac{1}{r} && (15)\\
A_5 &= A_3 + A_4 && (16)\\
A_5 - \frac{1}{\widehat{\mu}(r+\lambda)} &= A_3 - A_4 && (17)\\
\frac{A_2}{\widetilde{X}} + \frac{\overline{x} - \lambda P}{r} &= A_3\widehat{X} + \frac{A_4}{\widehat{X}} + \frac{\overline{x}}{r+\lambda} + \frac{\lambda A_1\overline{X}}{r(1 - s^2/\sigma^2) + \lambda} && (18)\\
-\frac{\widetilde{\mu}A_2}{\widetilde{X}} + \frac{1}{r} &= \widehat{\mu}A_3\widehat{X} - \frac{\widehat{\mu}A_4}{\widehat{X}} + \frac{1}{r+\lambda} + \frac{\mu\lambda A_1\overline{X}}{r(1 - s^2/\sigma^2) + \lambda} && (19)
\end{aligned}
$$

where $\overline{X} = e^{\mu\overline{x}}, \widetilde{X} = e^{\widetilde{\mu}\overline{x}}$, and $\widehat{X} = e^{\widehat{\mu}\overline{x}}$. We can then solve (14)-(19) for A_1, A_2, A_3, A_4, A_5, and $\overline{x}$.

From (14) and (15) we can obtain

$$
A_1\overline{X} = \frac{\widetilde{\mu}}{\mu + \widetilde{\mu}}\frac{\overline{x} - \lambda P}{r} - \frac{\widetilde{\mu}}{\mu + \widetilde{\mu}}P + \frac{1}{r(\mu + \widetilde{\mu})}. \tag{20}
$$

From (16) and (17) we can obtain

$$
A_4 = \frac{1}{2\widehat{\mu}(r+\lambda)}. \tag{21}
$$

Using (14) and (18) we can obtain

$$
A_3\widehat{X} = A_1\overline{X}\frac{r(1 - s^2/\sigma^2)}{r(1 - s^2/\sigma^2) + \lambda} + P - \frac{A_4}{\widehat{X}} - \frac{\overline{x}}{r+\lambda}. \tag{22}
$$

Using then (15), (20), (21), and (22) in (19) we can then obtain (5) in the paper, which determines $\overline{x}$.

Derivation of $\overline{x}$ in the Case of $\overline{x} < 0$:

Let $W(x)$ be the expected present value of payoffs for the consumer if the consumer does not own the product and is getting information on the product, $x < \overline{x} < 0$. Let $V(x)$ be the expected present value of payoffs for the consumer if the consumer owns the product and $x \geq \overline{x}$. Let $\widetilde{V}(x)$ be the expected value of payoffs for the consumer if the consumer owns the product and $x < \overline{x}$.

When the consumer does not own the product and is searching for information we can obtain that the evolution of $W(x)$ is characterized by

$$W(x) = e^{-r\,dt} E[W(x + dx)]. \tag{23}$$

Using Itô's Lemma, we can get $rW(x) = W''(x)\frac{\sigma^2}{2}$, from which we can obtain

$$W(x) = D_1 e^{\mu x} + \widetilde{D}_1 e^{-\mu x} \tag{24}$$

where $\mu = \sqrt{2r/\sigma^2}$ and D_1 and $\widetilde{D}_1$ are constants to be determined. Note that $\lim_{x\to-\infty} W(x) = 0$, so we obtain $\widetilde{D}_1 = 0$.

When the consumer owns the product and $x \geq 0$, we have that the expected present value of consumer payoffs has to satisfy

$$V(x) = x\,dt + e^{-r\,dt}\lambda\,dt\{E[V(x + dx)] - P\} + e^{-r\,dt}(1 - \lambda\,dt)E[V(x + dx)] \tag{25}$$

Using Itô's Lemma, this reduces to $rV(x) = x - \lambda P + V''(x)\frac{s^2}{2}$. Solving this differential equation one obtains

$$V(x) = \widetilde{D}_2 e^{\widetilde{\mu}x} + D_2 e^{-\widetilde{\mu}x} + \frac{x - \lambda P}{r}, \tag{26}$$

where $\widetilde{\mu} = \sqrt{2r/s^2}$, and D_2 and $\widetilde{D}_2$ are constants to be determined. Note that $\lim_{x\to\infty}[V(x) - (x - \lambda P)/r] = 0$. We then have that $\widetilde{D}_2 = 0$.

Consider now that the consumer owns the product and $\overline{x} \leq x < 0$. In this region, the consumer would not use the product, but would repurchase if the product breaks down. The expected present value of consumer payoffs has to satisfy

$$V(x) = e^{-r\,dt}\lambda\,dt\{E[V(x + dx)] - P\} + e^{-r\,dt}(1 - \lambda\,dt)E[V(x + dx)]. \tag{27}$$

Using Itô's Lemma, and solving the resulting differential equation, this yields

$$V(x) = D_3 e^{\widetilde{\mu}x} + D_4 e^{-\widetilde{\mu}x} - \frac{\lambda}{r}P, \tag{28}$$

where $\widetilde{\mu} = \sqrt{2r/s^2}$, and D_3 and D_4 are constants to be determined.

Finally, for the case of $x < \overline{x}$, the expected present value of consumer payoffs has to satisfy

$$\widetilde{V}(x) = e^{-r\,dt}\lambda\,dt W(x) + e^{-r\,dt}(1-\lambda\,dt)E\widetilde{V}(x+dx) \tag{29}$$

which yields

$$\widetilde{V}(x) = D_5 e^{\widehat{\mu}x} + \widetilde{D}_5 e^{-\widehat{\mu}x} + \frac{\lambda D_1}{r(1-s^2/\sigma^2)+\lambda} e^{\mu x}, \tag{30}$$

where D_5 and $\widetilde{D}_5$ are constants to be determined. Noting that $\lim_{x\to-\infty}\widetilde{V}(x) = 0$, we obtain $\widetilde{D}_5 = 0$.

Value matching and smooth pasting at both $\overline{x}$ and 0, $W(\overline{x}) = V(\overline{x}) - P, W'(\overline{x}) = V'(\overline{x}), V(\overline{x}) = \widetilde{V}(\overline{x}), V'(\overline{x}) = \widetilde{V}'(\overline{x}), V(0^+) = V(0^-)$, and $V'(0^+) = V'(0^-)$ yields

$$\begin{aligned}
D_1\overline{X} &= D_3\widetilde{X} + \frac{D_4}{\widetilde{X}} - \frac{\lambda P}{r} - P & (31)\\
\mu D_1\overline{X} &= \widetilde{\mu}D_3\widetilde{X} - \widetilde{\mu}\frac{D_4}{\widetilde{X}} & (32)\\
D_3\widetilde{X} + \frac{D_4}{\widetilde{X}} - \frac{\lambda P}{r} &= D_5\widehat{X} + \frac{\lambda D_1\overline{X}}{r(1-s^2/\sigma^2)+\lambda} & (33)\\
D_3\widetilde{X} - \frac{D_4}{\widetilde{X}} &= \frac{\widehat{\mu}}{\widetilde{\mu}}D_5\widehat{X} + \frac{\mu}{\widetilde{\mu}}\frac{\lambda D_1\overline{X}}{r(1-s^2/\sigma^2)+\lambda} & (34)\\
D_3 + D_4 &= D_2 & (35)\\
D_3 - D_4 &= -D_2 + \frac{1}{r\widetilde{\mu}} & (36)
\end{aligned}$$

where $\overline{X} = e^{\mu\overline{x}}, \widetilde{X} = e^{\widetilde{\mu}\overline{x}}$, and $\widehat{X} = e^{\widehat{\mu}\overline{x}}$. We can then solve (31)-(36) for D_1, D_2, D_3, D_4, D_5, and $\overline{x}$.

From (35) and (36) we can obtain

$$D_3 = \frac{1}{2r\widetilde{\mu}} \tag{37}$$

From (31) and (33) we can obtain

$$\frac{r(1-s^2/\sigma^2)}{r(1-s^2/\sigma^2)+\lambda}D_1\overline{X} = D_5\widehat{X} + P \tag{38}$$

Using (32) and (34) we can obtain

$$\frac{r(1-s^2/\sigma^2)}{r(1-s^2/\sigma^2)+\lambda}D_1\overline{X} = \frac{\widehat{\mu}}{\mu}D_5\widehat{X} \tag{39}$$

Combining (38) and (39), we get

$$D_1\overline{X} = P\frac{\widehat{\mu}}{\widehat{\mu}-\mu}\frac{r(1-s^2/\sigma^2)}{r(1-s^2/\sigma^2)+\lambda} \tag{40}$$

From (31) and (32) we can obtain

$$D_1\overline{X} = \frac{1}{r(\mu+\widetilde{\mu}})\overline{X} - \frac{\widetilde{\mu}}{\mu+\widetilde{\mu}}\frac{\lambda+r}{r}P \tag{41}$$

Combining (40) and (41) we get the closed-form solution for $\overline{x}$ as in (7) in the paper:

$$e^{\mu\overline{x}} = P\left[\widetilde{\mu}(\lambda+r) + r\widehat{\mu}\frac{\mu+\widetilde{\mu}}{\widehat{\mu}-\mu}\frac{r(1-s^2/\sigma^2)}{r(1-s^2/\sigma^2)+\lambda}\right] \tag{42}$$

Derivation of the Expected Discount Factor until the Next Purchase:

Using $\lim_{x\to\infty}\widetilde{d}(x) = \frac{\lambda}{\lambda+r}$, and Itô's Lemma in (14) in the paper, and solving the resulting differential equation, yields

$$\widetilde{\delta}(x) = B_1e^{-\widehat{\mu}x} + \frac{\lambda}{\lambda+r} \tag{43}$$

where we recall that $\widehat{\mu} = \sqrt{2(r+\lambda)/s^2}$, and B_1 is a constant to be determined.

From (15) in the paper we can obtain the differential equation

$$(r+\lambda)\widetilde{\delta}(x) = \lambda e^{-\mu(\overline{x}-x)} + \frac{s^2}{2}\widetilde{\delta}''(x). \tag{44}$$

Note that as $x \to -\infty$, the expected discount factor until the next purchase approaches zero. Using this when solving the differential equation (44) yields

$$\widetilde{\delta}(x) = B_2e^{\widehat{\mu}x} + \frac{\lambda}{\lambda+r(1-s^2/\sigma^2)}e^{-\mu(\overline{x}-x)}, \tag{45}$$

where B_2 is a constant to be determined.

Value matching and smooth pasting at $\overline{x}$, $\widetilde{\delta}(\overline{x}^+) = \widetilde{\delta}(\overline{x}^-)$ and $\widetilde{\delta}'(\overline{x}^+) = \widetilde{\delta}'(\overline{x}^-)$, yields B_1 and B_2 below, which fully determines $\widetilde{\delta}(x)$, and therefore $\widetilde{T}(x)$.

$$B_1 = \frac{\lambda e^{\widehat{\mu}\overline{x}}}{2}\left[\frac{1-\mu/\widehat{\mu}}{\lambda+r(1-s^2/\sigma^2)} - \frac{1}{\lambda+r}\right] \tag{46}$$

$$B_2 = \frac{\lambda e^{-\widehat{\mu}\overline{x}}}{2}\left[\frac{1}{\lambda+r} - \frac{1+\mu/\widehat{\mu}}{\lambda+r(1-s^2/\sigma^2)}\right] \tag{47}$$

DERIVATION OF THE EXPECTED NUMBER OF PURCHASES:

Let $N(x)$ be the expected number of units purchased going forward given that the consumer starts at $x < \overline{x}$ and the consumer does not own the product. We have that $N(x)$ evolves over time as

$$N(x) = (1-\beta\, dt)EN(x+dx). \tag{48}$$

Note that $\lim_{x\to-\infty} N(x) = 0$, as the number of expected purchases going forward approaches zero, when the current utility of owning/using the product goes to negative infinity. Using this when solving for (48) yields

$$N(x) = C_1 e^{\eta x} \tag{49}$$

where $\eta = \sqrt{2\beta/\sigma^2}$, and C_1 ia a constant to be determined.

Let $\widetilde{N}(x)$ be the expected number of future units purchased over time given that the consumer owns the product. As the consumer purchases the product immediately if the consumer does not own the product and $x = \overline{x}$, we have

$$N(\overline{x}) = 1 + \widetilde{N}(\overline{x}). \tag{50}$$

For $x \geq \overline{x}$ the evolution of $\widetilde{N}(x)$ over time has to satisfy

$$\widetilde{N}(x) = \lambda\, dt\, [1+\widetilde{N}(x)] + (1-\lambda\, dt - \beta\, dt)E\widetilde{N}(x+dx). \tag{51}$$

Note also that $\lim_{x\to\infty} \widetilde{N}(x) = \lambda/\beta$. To see this, we can obtain that the expected duration of the consumer in the market is $1/\beta$, and the expected duration of the product is $1/\lambda$. Then, if the consumer always repurchased the product when it broke down, the consumer would make on "average" λ/β purchases going forward. As the current utiliy of owning the product approaches infinity, the consumer behaves as if always repurchasing the product when it breaks down, and therefore the expected number of purchases going forward, $\widetilde{N}(x)$

approaches λ/β. Using this, when solving for (51), yields

$$\widetilde{N}(x) = C_2 e^{-\widetilde{\eta}x} + \frac{\lambda}{\beta}, \tag{52}$$

where $\widetilde{\eta} = \sqrt{2\beta/s^2}$. and C_2 is a constant to be determined.

Consider now the evolution of $\widetilde{N}(x)$ for $x < \overline{x}$. This yields

$$\widetilde{N}(x) = \lambda\, dt\, N(x) + (1 - \lambda\, dt - \beta\, dt)E\widetilde{N}(x + dx). \tag{53}$$

By Itô's Lemma, this can be written as

$$(\beta + \lambda)\widetilde{N}(x) = \lambda C_1 e^{\eta x} + \frac{s^2}{2}\widetilde{N}''(x). \tag{54}$$

Note that $\lim_{x\to-\infty} \widetilde{N}(x) = 0$ as when the current utility x of owning the product approaches negative infinity, the consumer is expected not to make any more purchases going forward. Using this when solving for (54) yields

$$\widetilde{N}(x) = C_3 e^{\widehat{\eta}x} + C_1 \frac{\lambda}{\lambda + \beta(1 - s^2/\sigma^2)} e^{\eta x}, \tag{55}$$

where $\widehat{\eta} = \sqrt{2(\beta + \lambda)/s^2}$, and C_3 is a constant to be determined.

The value matching and smooth pasting conditions at $\overline{x}$ presented in the text yield

$$\frac{C_2}{\widetilde{Y}} + \frac{\lambda}{\beta} = C_3\widehat{Y} + C_1\overline{Y}\frac{\lambda}{\lambda + \beta(1 - s^2/\sigma^2)} \tag{56}$$

$$-\widetilde{\eta}\frac{C_2}{\widetilde{Y}} = \widehat{\eta}C_3\widehat{Y} + \eta C_1\overline{Y}\frac{\lambda}{\lambda + \beta(1 - s^2/\sigma^2)}. \tag{57}$$

where $\overline{Y} = e^{\eta\overline{x}}, \widetilde{Y} = e^{\widetilde{\eta}\overline{x}}$, and $\widehat{Y} = e^{\widehat{\eta}\overline{x}}$. Using (50), (56), and (57) we can obtain C_1, C_2, and C_3 as a function of $\overline{x}$.[3]

Using (50) in both (56) and (57) we can solve for $C_2/\widetilde{Y}$ to obtain

$$\frac{C_2}{\widetilde{Y}} = \frac{\widehat{\eta}s^2/\sigma^2 - \eta(\lambda + \beta)/\beta}{\widehat{\eta}\frac{\beta(1-s^2/\sigma^2)}{\lambda} + \widetilde{\eta}(\frac{\lambda+\beta(1-s^2/\sigma^2)}{\lambda}) + \eta}. \tag{58}$$

[3]Note that we do not have smooth pasting at $\overline{x}$ between $N(x)$ and $\widetilde{N}(x)$ as there is no optimality decision on the derivation of these functions. Note also that the smooth pasting condition for $\widetilde{N}(x)$ at $\overline{x}$ is not an optimality condition, but it is rather due to the infinite variation of the Brownian motion.

Using (50) we can then obtain (21) in the paper.

Proof of Proposition 3:

The comparative statics for the expected number of purchases going forward immediately after a purchase can be directly obtained from evaluating (22) in the paper at $\overline{x}, N(\overline{x})$. The comparative statics for λ and β are straightforward. To get the comparative statics with respect to the ratio s^2/σ^2, let $\varepsilon = \sqrt{s^2/\sigma^2}$ and $w = \sqrt{1+\lambda/\beta}$. Using (21) in the paper we can obtain

$$N(\overline{x}) = w^2 + \frac{\varepsilon^2\sqrt{\frac{2(\beta+\lambda)}{s^2}} - w^2\sqrt{\frac{2\beta}{\sigma^2}}}{\frac{\beta(1-\varepsilon^2)}{\lambda}\sqrt{\frac{2(\beta+\lambda)}{s^2}} + \frac{\lambda+\beta(1-\varepsilon^2)}{\lambda}\sqrt{\frac{2\beta}{s^2}} + \sqrt{\frac{2\beta}{\sigma^2}}}. \tag{59}$$

Multiplying the second term in the right hand side of (59) on the numerator and denominator by $\sqrt{\frac{s^2}{2\beta}}$ and then by $(w-1)$, we can then obtain, using (18) in the paper,

$$\widetilde{N}(\overline{x}) = w^2 - 1 + \frac{w(w-1)(\varepsilon^2 - w\varepsilon)}{1-\varepsilon^2+(w-1)(1+\varepsilon)}, \tag{60}$$

where the derivative to ε is negative.

The comparative statics with respect to $\beta, \widetilde{r}$, and P, of the expected number of purchases going forward after an initial current utility of $x < \overline{x}$ can be directly obtained by differentiating (22) in the paper. The comparative statics with respect to $s^2 = \sigma^2$ and λ require a little more analysis.

Consider first the comparative statics with respect to $s^2 = \sigma^2$. We can obtain:

$$\frac{\partial N(x)}{\partial \sigma^2}_{\sigma^2=s^2} = \left[\frac{\partial \eta}{\partial \sigma^2}(x-\overline{x}) - \eta\frac{\partial \overline{x}}{\partial \sigma^2}_{\sigma^2=s^2}\right] N(x). \tag{61}$$

As both $\frac{\partial \eta}{\partial \sigma^2}(x-\overline{x})$ and $\eta\frac{\partial \overline{x}}{\partial \sigma^2}_{\sigma^2=s^2}$ are positive, we can see that the size of $|x-\overline{x}|$ determines the sign of $\frac{\partial N(x)}{\partial \sigma^2}_{\sigma^2=s^2}$, which is positive (negative) is X is low (high) enough.

Consider now the comparative statics with respect to λ. We can obtain

$$\frac{\partial N(x)}{\partial \lambda} = \frac{1}{2}e^{\eta(x-\overline{x})}\left[\frac{1}{\beta}\left(1+\frac{1}{2}\sqrt{\frac{\beta}{\lambda+\beta}}\right) - \eta\frac{\partial \overline{x}}{\partial \lambda}\right], \tag{62}$$

which is negative if β is not too low.

Derivation of $G(x)$ and $\widetilde{G}(x)$:

We have that $G(x)$ evolves over time as

$$G(x) = e^{-r\,dt} E G(x + dx), \tag{63}$$

which yields

$$G(x) = C_1 e^{\mu x} + C_2 e^{-\mu x} \tag{64}$$

where C_1 and C_2 are constants to be determined. As $\lim_{x \to -\infty} G(x) = 0$, we have $C_2 = 0$.

Let $\widetilde{G}(x)$ be the discounted number of future units purchased over time given that the consumer owns the product. As the consumer purchases the product immediately if the consumer does not own the product and $x = \overline{x}$, we have

$$G(\overline{x}) = 1 + \widetilde{G}(\overline{x}). \tag{65}$$

For $x \geq \overline{x}$ the evolution of $\widetilde{G}(x)$ over time has to satisfy

$$\widetilde{G}(x) = \lambda\, dt\, [1 + \widetilde{G}(x)] + (1 - \lambda\, dt) e^{-r\,dt} E \widetilde{G}(x + dx), \tag{66}$$

which yields

$$\widetilde{G}(x) = C_3 e^{\widetilde{\mu} x} + C_4 e^{-\widetilde{\mu} x} + \frac{\lambda}{r}, \tag{67}$$

where C_3 and C_4 are constants to be determined. As $\lim_{x \to \infty} \widetilde{G}(x) = \lambda/r$, we have $C_3 = 0$.

Consider now the evolution of $\widetilde{G}(x)$ for $x < \overline{x}$. This yields

$$\widetilde{G}(x) = \lambda\, dt\, G(x) + (1 - \lambda\, dt) e^{-r\,dt} E \widetilde{G}(x + dx). \tag{68}$$

By Itô's Lemma, this can be written as

$$(r + \lambda)\widetilde{G}(x) = \lambda C_1 e^{\mu x} + \frac{s^2}{2} \widetilde{G}''(x), \tag{69}$$

which yields

$$\widetilde{G}(x) = C_5 e^{\widehat{\mu} x} + C_6 e^{-\widehat{\mu} x} + C_1 \frac{\lambda}{\lambda + r(1 - s^2/\sigma^2)} e^{\mu x}, \tag{70}$$

where C_5 and C_6 are constants to be determined. As $\lim_{x \to -\infty} \widetilde{G}(x) = 0$, we have $C_6 = 0$.

Using value matching and smooth pasting at $\overline{x}$ for $\widetilde{G}(x)$, $\widetilde{G}(\overline{x}^+) = \widetilde{G}(\overline{x}^-)$ and $\widetilde{G}'(\overline{x}^+) =$

$\widetilde{G}'(\overline{x}^-)$, yields

$$\frac{C_4}{\widetilde{X}} + \frac{\lambda}{r} = C_5\widehat{X} + C_1\overline{X}\frac{\lambda}{\lambda + r(1 - s^2/\sigma^2)} \tag{71}$$

$$-\widetilde{\mu}\frac{C_4}{\widetilde{X}} = \widehat{\mu}C_5\widehat{X} + \mu C_1\overline{X}\frac{\lambda}{\lambda + r(1 - s^2/\sigma^2)}. \tag{72}$$

Using (65), (71), and (72) we can obtain C_1, C_4, and C_5 as a function of $\overline{x}$.[4]

Using (65) in both (71) and (72) we can solve for $C_4/\widetilde{X}$ to obtain

$$\frac{C_4}{\widetilde{X}} = \frac{\widehat{\mu}s^2/\sigma^2 - \mu(\lambda + r)/r}{\widehat{\mu}\frac{r(1-s^2/\sigma^2)}{\lambda} + \widetilde{\mu}(\frac{\lambda+r(1-s^2/\sigma^2)}{\lambda}) + \mu}. \tag{73}$$

Using (65) we can then obtain (23) in the paper.

Proof of Proposition 4:

Given the presentation in the text, we have the characterization of the equilibrium as a function of x_0. To see that $\overline{x}$ is increasing in x_0 for $x_0 > x^{**}$ we can see that the right hand side of (31) in the paper is decreasing in $\overline{x}$ and increasing in x_0. To see that $\overline{x} < x_0$ for $x_0 > x^{**}$ we can just obtain that the total differentiation of (31) in the paper with respect to x_0 and $\overline{x}$ yields $\frac{\partial \overline{x}}{\partial x_0} < 1$.

For the comparative statics with respect to λ, r, and σ^2 (under the constraint $s^2 = \sigma^2$) let us consider each region of x_0 separately.

For $x_0 < x^*$, note that we can write (27) in the paper as

$$\overline{x} - \frac{\widehat{X} - 1}{\widehat{X}}\left(\frac{1}{\widehat{\mu}} + \frac{1}{\mu}\right) = 0. \tag{74}$$

The derivative of the left hand side with respect to $\overline{x}$ can be obtained to be $1 - a/(e^a - 1)$, after using (27) in the paper, where $a = \widehat{\mu}\overline{x}$. We can then obtain that that derivative is positive for $a > 0$. Taking the derivative of the left hand side of (74) with respect to $\widehat{\mu}$ we can obtain that it has the same sign of $(e^a - 1)^2 - a^2e^a$, which is positive for $a > 0$. Then, as $\widehat{\mu}$ is increasing in λ we can obtain $\frac{\partial \overline{x}}{\partial \lambda} < 0$.

[4]Note that we do not have smooth pasting at $\overline{x}$ between $G(x)$ and $\widetilde{G}(x)$ as there is no optimality decision on the derivation of these functions. Similarly to the case of $\widetilde{N}(x)$ above, note also that the smooth pasting condition for $\widetilde{G}(x)$ at $\overline{x}$ is not an optimality condition, but it is rather due to the infinite variation of the Brownian motion.

The derivative of the left hand side of (74) with respect to r is equal to the derivative with respect to λ, which we saw was positive, plus $\frac{\widehat{X}-1}{\widehat{X}}\frac{1}{\mu^2}\frac{\partial\mu}{\partial r} > 0$, which then yields $\frac{\partial\overline{x}}{\partial r} < 0$.

The derivative of the left hand side of (74) with respect to σ^2 under the constraint $\sigma^2 = s^2$ is composed with two terms, one though $\widehat{\mu}$ and the other through μ, where both are negative (the first one has the opposite sign of the derivative with respect to λ and the second is $\frac{\widehat{X}-1}{\widehat{X}}\frac{1}{\mu^2}\frac{\partial\mu}{\partial\sigma^2} < 0$). This then yields $\frac{\partial\overline{x}}{\partial\sigma^2}_{|s^2=\sigma^2} > 0$.

For the case when $s^2 \neq \sigma^2$ we can obtain from equation (5) in the paper

$$\frac{\partial\overline{x}}{\partial P} = \widehat{X}(r+\lambda)\frac{1-A}{\widehat{X}-1-A\widehat{X}}, \tag{75}$$

where $A = \frac{\widetilde{\mu}}{\widehat{\mu}}\frac{\mu-\widehat{\mu}}{\mu+\widetilde{\mu}}\frac{(r+\lambda)(s^2/\sigma^2-1)}{\lambda-r(s^2/\sigma^2-1)}$, and we can then obtain the optimal price using (24) in the paper as

$$P = \frac{1}{\mu(\lambda+r)\widehat{X}}\frac{\widehat{X}-1-A\widehat{X}}{1-A}. \tag{76}$$

As in the case of $s^2 = \sigma^2$ we can then obtain the equivalent to (74) by using (76) in (5) in the paper:

$$\overline{x} - \frac{\widehat{X}-1}{\widehat{X}}\left(\frac{1}{\widehat{\mu}}+\frac{1}{\mu}\right) - A\left(\overline{x}-\frac{1}{\mu}+\frac{1}{\widetilde{\mu}}\right) = 0, \tag{77}$$

Denoting the first two terms of (77) as $\phi_{12}(s^2)$ we have

$$\frac{\partial\phi_{12}}{\partial s^2} = -\frac{\widehat{X}-1-\widehat{\mu}\overline{x}(1+\widehat{\mu}/\mu)}{2s^2\widehat{\mu}\widehat{X}}, \tag{78}$$

as the derivative of the first two terms with respect to $\widehat{\mu}$ is $\frac{\widehat{X}-1-\widehat{\mu}\overline{x}(1+\widehat{\mu}/\mu)}{\widehat{\mu}^2\widehat{X}}$, and the derivative of $\widehat{\mu}$ with respect to s^2 is $-\frac{\widehat{\mu}}{2s^2}$. Denoting the third term of (77) as $\phi_3(s^2)$ we can obtain

$$\frac{\partial\phi_3}{\partial s^2} = \frac{\widetilde{\mu}(\widehat{\mu}-\mu)(r+\lambda)\overline{x}}{\sigma^2\lambda\widehat{\mu}(\mu+\widetilde{\mu})}, \tag{79}$$

when it is evaluated at $s^2 = \sigma^2$, given that $s^2/\sigma^2 - 1 = 0$ when $s^2 = \sigma^2$.

When $s^2 = \sigma^2$ we have from the above analysis of (74) that the left hand side of (77) is increasing in $\overline{x}$. Adding $\frac{\partial \phi_{12}}{\partial s^2}$ and $\frac{\partial \phi_3}{\partial s^2}$, and using (27) in the paper, we can obtain

$$\begin{aligned}
\text{Sign}\left\{\frac{\partial \overline{x}}{\partial s^2}\right\} &= \text{Sign}\left\{\frac{\widehat{X} - 1 - \widehat{\mu}\overline{x}(1 + \widehat{\mu}/\mu)}{2\widehat{X}} - \frac{\widetilde{\mu}(\widehat{\mu} - \mu)(r + \lambda)\overline{x}}{\lambda(\mu + \widetilde{\mu})}\right\} \\
&= \text{Sign}\Bigg\{\lambda(\widehat{X} - 1) - \lambda\widehat{\mu}\overline{x}\left(1 + \sqrt{\frac{r+\lambda}{r}}\right) - (r + \lambda)\left(1 + \sqrt{\frac{r+\lambda}{r}}\right)\Bigg(1 - \\
&\qquad \sqrt{\frac{r}{r+\lambda}}\Bigg)(\widehat{X} - 1)\Bigg\} \qquad (80)
\end{aligned}$$

when evaluated at $s^2 = \sigma^2$, which is negative as $(r+\lambda)(1+\sqrt{(r+\lambda)/r})(1-\sqrt{r/(r+\lambda)}) > \lambda$.

To check the effect on P we can use (74) in (6) in the paper to obtain

$$P = \overline{x}\left(\frac{1}{\lambda + r + \sqrt{r(\lambda + r)}}\right). \qquad (81)$$

when $s^2 = \sigma^2$. Given the results on the comparative statics on $\overline{x}$ we can then immediately obtain $\frac{\partial P}{\partial \lambda}, \frac{\partial P}{\partial r} < 0$, and $\frac{\partial P}{\partial \sigma^2}_{|s^2=\sigma^2} > 0$.

For the case in which $s^2 \neq \sigma^2$, we can similarly use (77) in (5) in the paper to obtain

$$P = \overline{x}\left(\frac{1}{\lambda + r + \sqrt{r(\lambda + r)}}\right) - \frac{A(1 + \widehat{\mu}/\widetilde{\mu})}{(r + \lambda)(1 - A)(\mu + \widehat{\mu})} \qquad (82)$$

As we have

$$\frac{\partial \overline{x}}{\partial s^2} = \frac{\lambda[e^a - 1 - a(1 + \gamma)] - (\gamma - 1/\gamma)(r + \lambda)(e^a - 1)}{2\lambda s^2 \widehat{\mu}\widehat{X}} \frac{e^a - 1}{e^a - 1 - a} \qquad (83)$$

where $a = \widehat{\mu}\overline{x}, \gamma = \widehat{\mu}/\mu$, when evaluated at $s^2 = \sigma^2$, we can obtain

$$\frac{\partial P}{\partial s^2} = \frac{\partial \overline{x}}{\partial s^2} \frac{\widehat{\mu}}{(r + \lambda)(\mu + \widehat{\mu})} + \frac{1 - \mu/\widehat{\mu}}{2\lambda\mu\sigma^2} \qquad (84)$$

from which we can obtain, using (27) in the paper

$$\text{Sign}\left\{\frac{\partial P}{\partial s^2}\right\} = \text{Sign}\{e^a - 1 - a - a^2\}. \qquad (85)$$

As we have that a is increasing in λ/r and $e^a - 1 - a - a^2 < 0$ for $\lambda = 0$ and $e^a - 1 - a - a^2 > 0$

for $\lambda/r \to \infty$, we then have that $\frac{\partial P}{\partial s^2} < 0$ for λ/r sufficiently small, and $\frac{\partial P}{\partial s^2} > 0$ for λ/r sufficiently large.

To obtain the equivalent to (31) in the paper, the equation $h(\overline{x}, x_0) = 0$, for when $s^2 \neq \sigma^2$, we can use (5) in the paper in (30) in the paper to yield

$$1 + \widetilde{G}(x_0) + \widetilde{\mu}\widehat{X}[\widetilde{G}(x_0) - \frac{\lambda}{r}] \frac{1-A}{\widehat{X} - 1 - A\widehat{X}} \left[\overline{x} + \frac{1-\widehat{X}}{\widehat{\mu}\widehat{X}} - (\overline{x} + 1/\widetilde{\mu}) \right] = 0. \tag{86}$$

As in the case of $s^2 = \sigma^2$, when $s^2 \neq \sigma^2$ we can obtain x^{**} by $h(x^{**}, x^{**}) = 0$.

For $x_0 \in [x^*, x^{**}]$, we can just use the derivation in Proposition 1 as the purchase threshold is fixed at x_0, and we can just compute the effect of λ, r, s^2 and σ^2, under the constraint of $s^2 = \sigma^2$, by just total differentiating (5) in the paper with respect to P and each of the variables under interest. We can then obtain that the optimal price is decreasing in λ, r, and σ^2, under the constraint of $s^2 = \sigma^2$, and increasing in s^2.

Consider now the case of $x_0 > x^{**}$. Consider first the effect of λ. We can obtain that the derivative of the right hand side of (31) in the paper, using also that expression, has the sign equal to the sign of $1 - C_1/2 - C_2/2$ where

$$C_1 = \sqrt{1+\frac{\lambda}{r}} \Big/ \left(\sqrt{1+\frac{\lambda}{r}} - 1 \right), \tag{87}$$

$$C_2 = \frac{\frac{ae^a}{e^a-1}\left(1 - \frac{a}{e^a-1}\right) + \sqrt{1+\frac{\lambda}{r}}}{\frac{ae^a}{e^a-1} - 1 + \sqrt{1+\frac{\lambda}{r}}}. \tag{88}$$

We can then obtain that $C_1, C_2 > 1$, so that this derivative is negative, which implies that $\overline{x}$ is decreasing in λ given that the left hand side of (31) in the paper is decreasing in $\overline{x}$. From (6) in the paper we can then obtain that P is also decreasing in λ given Proposition 1.

Consider now the effect of r. The derivative of the left hand side of (31) in the paper with respect to r can be obtained to be

$$-\frac{\lambda}{r^2} + \frac{\lambda}{2r^2} \frac{\sqrt{1+\frac{\lambda}{r}}}{\sqrt{1+\frac{\lambda}{r}} - 1} - \frac{1}{2r} \frac{\frac{ae^a}{e^a-1}\left(1 - \frac{a}{e^a-1}\right) - \frac{\lambda}{r}\sqrt{1+\frac{\lambda}{r}}}{\frac{ae^a}{e^a-1} - 1 + \sqrt{1+\frac{\lambda}{r}}} + \left(1 + \frac{\lambda}{r}\right)(x_0 - \overline{x}) \frac{1}{\sqrt{2rs^2}} \tag{89}$$

which can be either negative or positive. For example, it can be negative for $x_0 = x^{**}$ and λ large; it can be positive for λ small and x_0 large. Then, $\overline{x}$ can either increase or decrease with an increase in r.

Consider now the effect of σ^2, under the constraint $s^2 = \sigma^2$. The sign of the derivative of the left hand side of (31) in the paper can be obtained to be the same as the sign of

$$-\frac{ae^a}{e^a-1}\left(1-\frac{a}{e^a-1}\right)e^{\widetilde{\mu}(\overline{x}-x_0)}+2\widetilde{x}(x_0-\overline{x})\frac{1+\frac{\lambda}{r}}{\sqrt{1+\frac{\lambda}{r}}-1}, \tag{90}$$

which can be either positive or negative. For example, it can be positive if x_0 is sufficiently large; and it can be negative if $x_0 = x^{**}$.

Finally, consider the effect of s^2 on $\overline{x}$ while keeping σ^2 fixed for $x_0 > x^{**}$. The sign of the derivative of the left hand side of (86) with respect to s^2 can be obtained to be the same as the sign of

$$\begin{aligned}&\Big(1-\sqrt{\frac{r+\lambda}{r}}\Big)\widehat{X}\Big[-\widetilde{\mu}\overline{x}-\widetilde{\mu}\overline{x}\Big(\widetilde{\mu}\overline{x}-\sqrt{\frac{r}{r+\lambda}}\Big)-(\widetilde{\mu}\overline{x}+1)\Big(\sqrt{\frac{r}{r+\lambda}}-1\Big)\frac{r+\lambda}{\lambda}\Big]+\\&\Big[\widetilde{\mu}\overline{x}\widehat{X}+\sqrt{\frac{r}{r+\lambda}}(1-\widehat{X})\Big]\Big[-\Big(3+\sqrt{\frac{\lambda+r}{r}}\Big)/2-\widetilde{\mu}\Big(1-\sqrt{\frac{\lambda+r}{r}}\Big)(\overline{x}-x_0)-\Big(1-\sqrt{\frac{r}{r+\lambda}}\Big)^2\frac{r+\lambda}{\lambda}\Big]\end{aligned} \tag{91}$$

when evaluated at $s^2 = \sigma^2$, which can be either positive or negative depending on the values of x_0 and λ.

Derivation of the Analysis Related to Proposition 5 When $s^2 \neq \sigma^2$:

The optimal price is determined by (76), and the discounted number of purchases is given by (23) in the paper. Then, the firm's expected profit $\Pi(x_0) = PG(x_0)$ is

$$\Pi(x_0) = \frac{D}{\mu(\lambda+r)\widehat{X}}\frac{\widehat{X}-1-\widetilde{A}\widehat{X}}{1-\widetilde{A}}e^{\mu(x_0-\overline{x})}, \tag{92}$$

where

$$\begin{aligned}\widetilde{A} &= -\frac{s^2/\sigma^2-1}{s^2/\sigma^2+1}\frac{\sqrt{\frac{r+\lambda}{r}}}{\sqrt{\frac{r+\lambda}{r}}+\sqrt{s^2/\sigma^2}}\\ D &= \frac{\sqrt{\frac{r+\lambda}{r}}(\frac{r+\lambda}{r}-s^2/\sigma^2)(1+\sqrt{\frac{r+\lambda}{r}})}{(1+\sqrt{s^2/\sigma^2})(\frac{r+\lambda}{r}-1)+(1-s^2/\sigma^2)(1+\sqrt{\frac{r+\lambda}{r}})}.\end{aligned} \tag{93}$$

Given the analysis in the main text, we know that for s^2 close to σ^2 that the firm's expected profit increases in λ.

What happens at the limit of $\lambda \to \infty$? From (77) we get that $\overline{x} \to \frac{2}{\mu} - \frac{1}{\widetilde{\mu}}$ as $\lambda \to \infty$. Then from (76) we get $P \to \frac{1}{\mu}$ as $\lambda \to \infty$. Thus at the limit, the expected profit becomes

$$\Pi(x_0) = \frac{e^{\mu x_0 - 2 + \mu/\widetilde{\mu}}}{r\mu(1 + \sqrt{s^2/\sigma^2})} \tag{94}$$

Sketch of the Proof That Expected Time until the Next Purchase Is Infinity: This is related to the Gambler's Ruin problem. Consider a discrete random walk in the set of natural numbers, going up $+1$ or -1 with equal probability. What is the expected time to reach zero given that the process starts at 1? Let $\overline{T}(n)$ be the expected time for the process to reach zero when starting at n. We have

$$\overline{T}(1) = \frac{1}{2} + \frac{1}{2}\overline{T}(2). \tag{95}$$

Given spatial homogeneity, we know that expected time to go to 1 starting from 2 is $\overline{T}(1)$, and therefore $\overline{T}(2) = 2\overline{T}(1)$. Substituting in (95) we obtain $\overline{T}(1) = \frac{1}{2} + \overline{T}(1)$ which is only true for $\overline{T}(1) = \infty$.

In the continuous case considered here, let $\overline{T}(x)$ be the expected time until the first purchase for $x < \overline{x}$. If $\overline{T}(x)$ is bounded we can obtain $\overline{T}(x) = dt + E\overline{T}(x + dx)$ which leads to the second order differential equation $\overline{T}''(x) = -2/\sigma^2$. This then leads to $\lim_{x\to-\infty} \overline{T}(x) = -\infty$, which is not possible. Then, $\overline{T}(x)$ is not bounded for $x < \overline{x}$.

Derivation of Case with Returns at the Purchase Price P: When returns are always possible at the purchase price P, then the consumer's purchase threshold and return threshold are the same. The consumer buys the product if the expected current valuation x reaches $\overline{x}$, and returns the product when x dips below $\overline{x}$. Use the value function for $x < \overline{x}$ from (x), $W(x)$, the value function for $x > \overline{x}$ from (xi), $V(x)$, and the boundary conditions $W(x) = V(x) - P$ and $W'(x) = V'(x)$, we get

$$W(x) = \frac{1}{1 + \mu/\widetilde{\mu}} \left(\frac{\overline{x} - \lambda P}{r} + \frac{1}{\mu r} - P \right) e^{-\mu\overline{x}} e^{\mu x} \tag{96}$$

$$V(x) = \frac{1}{1 + \widetilde{\mu}/\mu} \left(-\frac{\overline{x} - \lambda P}{r} + \frac{1}{\mu r} + P \right) e^{\mu\overline{x}} e^{\mu x} \tag{97}$$

From which we see that $W(x)$ and $V(x)$ are maximized at $\overline{x} = (r+\lambda)P + 1/\mu - 1/\widetilde{\mu}$.

To derive the optimal price and product duration for the case of $s^2 = \sigma^2$ we can obtain the present value of profits when the consumer does not own the product as $\Pi(x_0)$ as follows. For $x < \overline{x}$ we have

$$\Pi(x) = e^{-r\,dt} E\Pi(x+dx), \tag{98}$$

which leads to $\Pi(x) = F_1 e^{\mu x}$, given that $\lim_{x\to-\infty} \Pi(x) = 0$, where F_1 is a constant to be determined.

For $x > \overline{x}$ we have the present value of value of profits going forward when the consumer owns the product as

$$\widetilde{\Pi}(x) = P\lambda\, dt + e^{-r\,dt} E\widetilde{\Pi}(x+dx), \tag{99}$$

which leads to $\widetilde{\pi}(x) = F_2 e^{-\widetilde{\mu}x} + \lambda P/r$, given that $\lim_{x\to\infty} \widetilde{\Pi}(x) = \lambda P/r$, where F_2 is a constant to be determined. The value matching and smooth pasting conditions, $\Pi(\overline{x}) + P = \widetilde{\Pi}(\overline{x})$ and $\Pi'(\overline{x}) = \widetilde{\Pi}'(\overline{x})$, leads to

$$F_1 = \frac{\widetilde{\mu}}{\mu+\widetilde{\mu}} P(\lambda/r - 1) e^{-\mu(r+\lambda)P} \tag{100}$$

$$F_2 = -\frac{\mu}{\mu+\widetilde{\mu}} P(\lambda/r - 1) e^{\widetilde{mu}(r+\lambda)P}. \tag{101}$$

We can then obtain $\Pi(x) = \frac{\widetilde{\mu}}{\mu+\widetilde{\mu}} P(\lambda/r - 1) e^{\mu[x-(r+\lambda)P]}$, from which we can obtain $P = \frac{1}{\mu(r+\lambda)}$ and that an infinitely small product duration is optimal. At the limit, the problem converges to the case without product returns, as an infinitely short duration makes return irrelevant.

Derivation of Dynamic Pricing:

Let us look for two prices, an initial price before the first purchase, P_0, and subsequent price after the first purchase, P_1. We denote the threshold for the first purchase with P_0 as $\overline{x}_0$, and denote the threshold for the subsequent purchases with P_1 as $\overline{x}_1$. We consider the case with $s^2 = \sigma^2$ so that $\overline{x}_0 > 0$ and $\overline{x}_1 > 0$.

There are two possibilities at time 0. If the consumer does not buy at time 0, then we must have $\overline{x}_0 > x_0$. If the consumer buys at time 0, then $\overline{x}_0 \le x_0$. However, if $\overline{x}_0 < x_0$, then we must still have $\overline{x}_0 < x_0$ for a marginal increase in P_0, which strictly increases the firm's profit as the consumer immediately buys under a slightly higher initial price, with the same expected profit after the initial purchase. Thus under optimal pricing, we must have $\overline{x}_0 \ge x_0$.

From equation (6), we can write the subsequent price, P_1, as a function of $\overline{x}_1$:

$$P_1 = \frac{\widehat{\mu}\overline{x}_1 + e^{-\widehat{\mu}\overline{x}_1} - 1}{\widehat{\mu}(r+\lambda)}. \tag{102}$$

After the first purchase, the value functions presented in the paper continue to hold. So, we can focus the value function under dynamic pricing for the consumer before the first purchase, $\widetilde{W}(x)$.

The Bellman equation for $\widetilde{W}(x)$ is

$$\widetilde{W}(x) = e^{-r\,dt} E\widetilde{W}(x+dx), \tag{103}$$

which leads to the solution,

$$\widetilde{W}(x) = \widetilde{A}_1 e^{\mu x}. \tag{104}$$

Consider the problem of the firm. Because under optimal pricing we must have $\overline{x}_0 \geq x_0$, the expected profit for the firm at time 0, using the expected time until next purchase, $\delta(x_0)$, can be written as

$$\max_{P_0,P_1} e^{-\mu(\overline{x}_0 - x_0)}[P_0 + P_1\widetilde{G}(\overline{x}_0)]. \tag{105}$$

We already have P_1 as a function of $\overline{x}_1$. We can express P_0 and $\widetilde{G}(\overline{x}_0)$ as functions of $\overline{x}_0$ and $\overline{x}_1$, then have firm maximize over $\overline{x}_0$ and $\overline{x}_1$:

$$\max_{\overline{x}_0 \geq x_0, \overline{x}_1} e^{-\mu(\overline{x}_0 - x_0)}[P_0 + P_1\widetilde{G}(\overline{x}_0)]. \tag{106}$$

Case 1: $\overline{x}_0 \geq \overline{x}_1$

Now we solve for P_0 as a function of $\overline{x}_0$ and $\overline{x}_1$. The value matching and smooth pasting conditions at $\overline{x}_0$ are

$$\widetilde{W}(\overline{x}_0) = V(\overline{x}_0) - P_0 \tag{107}$$

$$\widetilde{W}'(\overline{x}_0) = V'(\overline{x}_0). \tag{108}$$

These conditions can be written as

$$\widetilde{A}_1 e^{\mu\overline{x}_0} = A_2 e^{-\widetilde{\mu}\overline{x}_0} + \frac{\overline{x}_0 - \lambda P_1}{r} - P_0 \tag{109}$$

$$\widetilde{A}_1 e^{\mu\overline{x}_0} = -\frac{\widetilde{\mu}}{\mu} A_2 e^{-\widetilde{\mu}\overline{x}_0} + \frac{1}{r\mu}, \tag{110}$$

from which we can obtain

$$P_0 = 2A_2 e^{-\widetilde{\mu}\overline{x}_0} + \frac{\overline{x}_0 - \lambda P_1}{r} - \frac{1}{r\mu}. \tag{111}$$

Note also that we can obtain A_2 as a function of $\overline{x}_1$ using (xx), and (xiv).

For $\overline{x_0} \geq \overline{x}_1$, from (lxx) we have

$$\widetilde{G}(\overline{x}_0) = C_4 e^{-\widetilde{\mu}\overline{x}_0} + \frac{\lambda}{r} \tag{112}$$

where C_4 is a function of $\overline{x}_1$ using (lxv), (lxxi), and (lxxii).

Case 2: $\overline{x}_0 < \overline{x}_1$

Now we solve for P_0 as a function of $\overline{x}_0$ and $\overline{x}_1$. The value matching and smooth pasting conditions at $\overline{x}_0$ are

$$\begin{aligned} \widetilde{W}(\overline{x}_0) &= \widetilde{V}(\overline{x}_0) - P_0 && (113) \\ \widetilde{W}'(\overline{x}_0) &= \widetilde{V}'(\overline{x}_0). && (114) \end{aligned}$$

These conditions can be written as

$$\begin{aligned} \widetilde{A}_1 e^{\mu\overline{x}_0} &= A_3 e^{\widehat{\mu}\overline{x}_0} + A_4 e^{-\widehat{\mu}\overline{x}_0} + \frac{\overline{x}_0}{r+\lambda} + A_1 e^{\mu\overline{x}_0} - P_0 && (115) \\ \widetilde{A}_1 e^{\mu\overline{x}_0} &= \frac{\widehat{\mu}}{\mu} A_3 e^{\widehat{\mu}\overline{x}_0} - \frac{\widehat{\mu}}{\mu} A_4 e^{-\widehat{\mu}\overline{x}_0} + \frac{1}{\mu(r+\lambda)} + A_1 e^{\mu\overline{x}_0} && (116) \end{aligned}$$

from which we can obtain

$$P_0 = (1 - \frac{\widehat{\mu}}{\mu}) A_3 e^{\widehat{\mu}\overline{x}_0} + (1 + \frac{\widehat{\mu}}{\mu}) A_4 e^{-\widehat{\mu}\overline{x}_0} + \frac{\overline{x}_0 - 1/\mu}{r+\lambda} \tag{117}$$

where A_3 and A_4 are functions of $\overline{x}_1$ using (xx), (xxi), and (xxii).

For $\overline{x}_0 < \overline{x}_1$, from (lxx) we have

$$\widetilde{G}(\overline{x}_0) = C_5 e^{\widehat{\mu}\overline{x}_0} + C_1 e^{\mu\overline{x}_0} \tag{118}$$

where C_1 and C_5 are functions of $\overline{x}_1$ using (lxv), (lxxi), and (lxxii).

For the constants, we have

$$C_4 = \widetilde{X}_1 \frac{\widehat{\mu} - \mu(\lambda + r)/r}{\widetilde{\mu} + \mu} \tag{119}$$

$$C_1 = \frac{(1 + \widetilde{\mu}/\widehat{\mu})C_4/\widetilde{X}_1 + \lambda/r}{(1 - \mu/\widehat{\mu})\overline{X}_1} \tag{120}$$

$$C_5 = \frac{1}{\widehat{X}_1}\left(\frac{C_4}{\widetilde{X}_1} + \frac{\lambda}{r} - C_1\overline{X}_1\right) \tag{121}$$

$$A_1 = \frac{1}{\overline{X}_1}\left(\frac{1}{2}\frac{\overline{x}_1 - \lambda P_1}{r} - \frac{1}{2}P_1 + \frac{1}{2r\mu}\right) \tag{122}$$

$$A_2 = \widetilde{X}_1\left(A_1\overline{X}_1 - \frac{\overline{x}_1 - \lambda P_1}{r} + P_1\right) \tag{123}$$

$$A_4 = \frac{1}{2\widehat{\mu}(r + \lambda)} \tag{124}$$

$$A_3 = \frac{1}{\widehat{X}_1}\left(P - \frac{A_4}{\widehat{X}_1} - \frac{\overline{x}_1}{r + \lambda}\right) \tag{125}$$

where $\overline{X}_1 = e^{\mu\overline{x}_1}$, $\widetilde{X}_1 = e^{\widetilde{\mu}\overline{x}_1}$, and $\widehat{X}_1 = e^{\widehat{\mu}\overline{x}_1}$.